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The impact of advertising in a duopoly model

Lambert Schoonbeek* and Peter Kooreman

Department of Economics, University of Groningen, The Netherlands

Summary. We investigate the impact of advertising in a simple static differentiated duopoly model. First, we consider the Nash equilibrium of the situation in which the duopolistic firms compete simultaneously with two instruments, i.e. the prices and the advertising expenditures. Second, we examine the Nash equilibrium of the situation in which the firms only compete in prices and do not advertise at all. Next, we compare the two different Nash equilibria in order to assess the impact of advertising. In particular, we characterize in terms of the model parameters the circumstances in which the profits, outputs and/or prices of each firm are greater (smaller) in the Nash equilibrium with advertising than in the Nash equilibrium without advertising. We show that the results depend on (a) the size of the (positive) effect of advertising of a firm on its own demand, (b) the size and nature (stimulating or adverse) of the cross-effect of the advertising of each firm on the demand of the other firm, and (c) the size of the autonomous demand of the firms.

Keywords: Duopoly, Advertising, Profits.

JEL Classification Numbers: D43, L13, M37.

* Corresponding address: L. Schoonbeek, Department of Economics, University of Groningen, P.O. Box 800, NL-9700 AV Groningen, The Netherlands. Phone: + 31 50 363 3798, Fax: + 31 50 363 7337. E-mail: L.Schoonbeek@eco.rug.nl.

1. Introduction

Advertising is common practice in our world. In almost every branch of industry, firms compete by using advertisements in order to promote their products. It is therefore natural that economists are interested in understanding the impact of advertising on profits, outputs and prices of firms. We recall that one of the well-known approaches to analyse the impact of advertising on the profits of a firm is in terms of a prisoner's dilemma game. In that case a simple duopolistic market game is considered in which each of the two firms has two possible choices: either it advertises or it does not advertise. Generally, it is simply *assumed* that the corresponding numerical 2×2 profit (payoff) matrix is such that for each firm the profit is higher if both firms do not advertise than if both firms do advertise. For recent examples, see Bierman and Fernandez (1998, p. 11), Nicholson (1995, p. 679) and Waldman and Jensen (1997, p. 324). We stress that in these cases the size of the relevant profits is not derived from an explicit model.

Motivated by the latter observation, we examine in this paper the impact of advertising in a model of a static duopoly with product differentiation. In particular, we derive and compare for all possible values of the model parameters the size of the profits, outputs (demands) and prices of each firm in (i) the Nash equilibrium if both firms simultaneously compete with each other in prices as well as in advertising expenditures, and (ii) the Nash equilibrium if both firms only compete in prices and there is no advertising. In this way, our analysis makes explicit in which circumstances the profits, outputs and/or prices are higher or smaller in case (i) than in case (ii). We notice that *a priori* the impact of advertising in our duopoly is not obvious. Take e.g. the comparison of the demands of the firms in case (i) and case (ii). On the one side, we assume in our model that advertising of a firm always has a positive effect on its own demand. On the other side, intuitively speaking, the presence of advertising might also lead to a higher price of this firm, which induces a negative effect on its demand. Furthermore, the situation is even more complicated since the advertising and price level chosen by a firm also have cross-effects on its rival's demand, and *vice versa*; recall that the firms are involved in a duopolistic game. In particular, we remark that advertising of one firm can have a stimulating or an adverse cross-effect on the demand of the other firm. Both kinds of effects are allowed in our analysis. Summarizing, we conclude that a more detailed analysis is needed in order to assess the ultimate effect on the demands of the firms. The same applies to the profits and prices.

We further remark that our analysis can be relevant for situations in which governments (contemplate to) prohibit advertising by firms in a specific industry. For example, in a growing number of countries firms in the cigarette industry are not allowed

to advertise for reasons of population health. In those situations it is interesting to know the impact of the prohibition of advertising on the profits, outputs and prices of the firms involved; see also Von Hofmann (1987). In particular, it is interesting to know whether it is possible that such a measure is also better, i.e. more profitable, for the firms themselves, whereas at the same time the outputs fall. In those cases the interests of the firms and the government coincide.

As the starting point, we take in Section 2 the duopoly model also used by Gasmi and Vuong (1991), Gasmi et al. (1992) and Kadiyali (1996). The straightforward and relatively simple structure of this model allows us to obtain unambiguous and intuitively appealing conclusions. First, we examine the Nash equilibrium of the case in which both firms simultaneously compete with each other in prices as well as in advertising expenditures. Extending the analysis of the three mentioned studies, we present a number of assumptions that are needed in order to guarantee that the Nash equilibrium is well defined. Next, we discuss the Nash equilibrium associated with the case in which the two duopolists do not advertise and only compete in prices. In Section 3 we compare in detail the profits of the firms in the Nash equilibrium with advertising and the Nash equilibrium without advertising. We present a characterization of all possible cases. It turns out that the cases can be classified according to (a) the size of the (positive) effect of advertising of a firm on its own demand, (b) the size and kind of the cross-effects of advertising on the rival's demand, and (c) the size of the autonomous demand of the firms (i.e. the constant term in the demand functions of the firms). Section 4 briefly discusses the comparison of the outputs and prices of the firms in the two Nash equilibria. We end up in Section 5.

We remark that the model used in this paper is a static game, i.e. no time is involved. Of course, one could also wish to take into account intertemporal effects of advertising. To do that, one could specify a two-period game in which the firms choose their advertising expenditures in the first period and subsequently choose their prices in the second period, cf. Schmalensee (1983). However, many studies in marketing have found that advertising effects upon demand depreciate very rapidly, see e.g. Clarke (1976) and Lilien, Kotler and Moorthy (1992, chapter 6) for a general discussion of this point. Kadiyali (1996, p. 455) also observes that many studies have found no "carry-over" effects of advertising beyond one quarter for nondurables. Therefore, our static formulation seems to be appropriate.

To conclude, we remark that related but different theoretical research also has investigated advertising and price decisions of firms. We mention a number of the seminal studies. First, Bagwell and Ramey (1994a,b) present models in which advertising directs uninformed consumers to the firms that offer better deals, i.e. lower prices. They show that the case in which the firms advertise is preferable in terms of social welfare

to the case in which they do not advertise. Second, Bagwell and Ramey (1988, 1990) analyse signalling games in which an incumbent firm can signal with its advertising and its price to deter or accomodate entry of a potential entrant who is uncertain about the costs or demand conditions in the market. Third, Milgrom and Roberts (1986) examine signaling games in which firms use advertising and prices to signal product quality to the consumers. Important in these studies is the assumption that there is some kind of asymmetric information present in the model. We do not make such an assumption in the present paper, however. For comprehensive reference works on marketing, see Eliashberg and Lilien (1993) and Lilien, Kotler and Moorthy (1992).

2. The duopoly model

We consider a duopoly with the demand functions

$$q_i = \gamma_{i0} + \alpha_{ii}p_i + \alpha_{ij}p_j + \gamma_{ii}A_i^{\frac{1}{2}} + \gamma_{ij}A_j^{\frac{1}{2}}, \quad i, j = 1, 2 \ (j \neq i), \quad (1)$$

where q_i , p_i and A_i represent, respectively, the output, price and advertising expenditures of firm i , and p_j is the price of firm j . The parameter γ_{i0} denotes the autonomous demand of firm i . We remark that (1) involves the reasonable assumptions that the demand is linear in prices – which is common in the literature – and that the marginal effect of extra advertising is positive but with a diminishing rate. More generally, one might possibly prefer to use terms of the type $A_i^{\delta_i}$ and $A_j^{\delta_j}$ in the demand function, with $0 < \delta_i, \delta_j < 1$. However, the choice $\delta_i = \delta_j = \frac{1}{2}$ is convenient for expositional purposes.

The cost function of firm i is given by

$$C_i(q_i) = c_i q_i, \quad i = 1, 2. \quad (2)$$

The parameters of (1) and (2) satisfy the following assumption:

Assumption 1 *We have:*

- (a) $\alpha_{ii} < 0$, $\alpha_{ij} > 0$, $\gamma_{i0} > 0$, $\gamma_{ii} > 0$, $c_i > 0$, $i, j = 1, 2 \ (j \neq i)$
- (b) $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} > 0$
- (c) $\alpha_{ii} + \frac{1}{4}\gamma_{ii}^2 < 0$, $i = 1, 2$.

The interpretation of (a) of Assumption 1 is straightforward. Note that $\alpha_{ij} > 0$ means that the goods of both firms are substitutes. Intuitively speaking, (b) means that taken together the ‘own-price’ effects dominate the ‘cross-price’ effects, i.e. the product $\alpha_{11}\alpha_{22}$ of the ‘own-price’ coefficients is larger than the product $\alpha_{12}\alpha_{21}$ of the ‘cross-price’ coefficients. The meaning of (c) will be discussed shortly. Here we only no-

tice that given the value of α_{ii} , (c) implies an upperbound on the size of the ‘own-advertising’ coefficient γ_{ii} , i.e. $\gamma_{ii} < 2\sqrt{-\alpha_{ii}}$. Remark further that we do not impose a restriction on the sign of the coefficients of the cross-effects of advertising. Advertising of firm j ($j \neq i$) has a stimulating effect on the demand of firm i if $\gamma_{ij} > 0$, and an adverse effect if $\gamma_{ij} < 0$. In general, the firms might be able to choose the nature of their advertising, but that possibility is disregarded here.

Using (1) and (2) we write the profit function of firm i as

$$\pi_i(p_i, p_j, A_i, A_j) = (p_i - c_i)(\gamma_{i0} + \alpha_{ii}p_i + \alpha_{ij}p_j + \gamma_{ii}A_i^{\frac{1}{2}} + \gamma_{ij}A_j^{\frac{1}{2}}) - A_i, \quad i, j = 1, 2 \ (j \neq i). \quad (3)$$

We consider the duopoly as a noncooperative game in which p_i and A_i are the decision variables of firm i and $\pi_i(p_i, p_j, A_i, A_j)$ is its payoff function ($i = 1, 2$). The first-order conditions associated with an interior optimum of the profit-maximization problem of firm i read

$$\frac{\partial \pi_i}{\partial p_i} = \gamma_{i0} + 2\alpha_{ii}p_i + \alpha_{ij}p_j + \gamma_{ii}A_i^{\frac{1}{2}} + \gamma_{ij}A_j^{\frac{1}{2}} - \alpha_{ii}c_i = 0 \quad (4)$$

$$\frac{\partial \pi_i}{\partial A_i} = \frac{1}{2}(p_i - c_i)\gamma_{ii}A_i^{-\frac{1}{2}} - 1 = 0. \quad (5)$$

Notice that (5) requires that $p_i - c_i > 0$. Further, (5) can be rewritten as $A_i = A_i(p_i)$. The second-order conditions must hold as well, i.e. we require that in the optimum the Hessian of $\pi_i(p_i, p_j, A_i, A_j)$ as a function of p_i and A_i is a negative definite matrix. It can be verified by using (5) that this requires that $\alpha_{ii} < 0$ and $\alpha_{ii} + \frac{1}{4}\gamma_{ii}^2 < 0$. These two inequalities are guaranteed by, respectively, (a) and (c) of Assumption 1. We finally observe that (5) implies that in the optimum the ratio of the advertising expenditures to the total revenues equals the product of the price-cost margin with the elasticity of demand with respect to advertising expenditures. This relates our analysis to the well-known Dorfman-Steiner condition, see e.g. Waterson (1984, pp. 128-134).

Now, let $(p_1^*, p_2^*, A_1^*, A_2^*)$ denote an (interior) Nash equilibrium of the game. The associated output and profit level of firm $i = 1, 2$ are denoted by $q_i^* = q_i(p_i^*, p_j^*, A_i^*, A_j^*)$ and $\pi_i^* = \pi_i(p_i^*, p_j^*, A_i^*, A_j^*)$. Using the set of first-order conditions of both firms it follows that we must have $R(p^* - c) = d$, where $p^* = (p_1^*, p_2^*)'$, $c = (c_1, c_2)'$, $d = (d_1, d_2)'$ with $d_i = -\gamma_{i0} - \alpha_{ii}c_i - \alpha_{ij}c_j$ ($i, j = 1, 2; j \neq i$), and matrix $R = (r_{ij})$ is given by

$$R = \begin{pmatrix} 2\alpha_{11} + \frac{1}{2}\gamma_{11}^2 & \alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22} \\ \alpha_{21} + \frac{1}{2}\gamma_{21}\gamma_{11} & 2\alpha_{22} + \frac{1}{2}\gamma_{22}^2 \end{pmatrix}. \quad (6)$$

Part (c) of Assumption 1 means that the diagonal elements of R are negative.

In order to guarantee that the Nash equilibrium is well defined – i.e. $p_i^* - c_i > 0$, $q_i^* > 0$, and $\pi_i^* > 0$ for $i = 1, 2$ – we present the following assumption regarding the signs of the elements of vector d , the determinant of matrix R and the off-diagonal elements of R :

Assumption 2 *We have:*

- (a) $\gamma_{i0} + \alpha_{ii}c_i + \alpha_{ij}c_j > 0$, $i, j = 1, 2$ ($j \neq i$)
- (b) $\det(R) = (2\alpha_{11} + \frac{1}{2}\gamma_{11}^2)(2\alpha_{22} + \frac{1}{2}\gamma_{22}^2) - (\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{21}\gamma_{11}) > 0$
- (c) $\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj} > 0$, $i, j = 1, 2$ ($j \neq i$).

Part (a) of Assumption 2 states the reasonable requirement that the demand of each firm is positive if both firms set their prices equal to their own marginal cost (i.e. the lowest possible value) and moreover both firms do not advertise. Note that part (a) means that $d_1 < 0$ and $d_2 < 0$. Next, (b) of Assumption 2 implies that R is a nonsingular matrix. As a result $p^* - c = R^{-1}d$. Using (c) of Assumption 1 and (b) and (c) of Assumption 2 it follows that the elements of matrix R^{-1} are negative. Together with (a) of Assumption 2, this guarantees that $p_i^* - c_i > 0$. The output level q_i^* equals

$$\begin{aligned} q_i^* &= \gamma_{i0} + \alpha_{ii}p_i^* + \alpha_{ij}p_j^* + \gamma_{ii}(A_i^*)^{\frac{1}{2}} + \gamma_{ij}(A_j^*)^{\frac{1}{2}} \\ &= -(p_i^* - c_i)\alpha_{ii}, \end{aligned} \quad (7)$$

where the last equality follows from (4). We see that $q_i^* > 0$ because $p_i^* - c_i > 0$. The profit level π_i^* reads

$$\begin{aligned} \pi_i^* &= (p_i^* - c_i) \left(\gamma_{i0} + \alpha_{ii}p_i^* + \alpha_{ij}p_j^* + \gamma_{ii}(A_i^*)^{\frac{1}{2}} + \gamma_{ij}(A_j^*)^{\frac{1}{2}} \right) - A_i^* \\ &= (p_i^* - c_i)(-\alpha_{ii}p_i^* + \alpha_{ii}c_i) - A_i^* \\ &= -(p_i^* - c_i)^2(\alpha_{ii} + \frac{1}{4}\gamma_{ii}^2), \end{aligned} \quad (8)$$

where the second and third equality follow, respectively, from (4) and (5). Part (c) of Assumption 1 implies that $\pi_i^* > 0$.

In order to discuss (b) and (c) of Assumption 2 somewhat further, examine the comparative statics effects of a (marginal) increase in the autonomous demand γ_{i0} for good i on the prices p_i^* and p_j^* , $i, j = 1, 2$ ($j \neq i$). It follows from $p^* - c = R^{-1}d$ that $\partial p_i^* / \partial \gamma_{i0} = -r_{jj} / \det(R)$ and $\partial p_j^* / \partial \gamma_{i0} = -r_{ji} / \det(R)$. In the comparative statics literature it is standard to suppose now that R is a stable matrix. This is tantamount to assuming that a naive dynamic adjustment process of the prices that results as a consequence of the (marginal) change in γ_{i0} is locally stable around the Nash equi-

librium, see e.g. Varian (1992, pp. 288, 289).¹ We recall that matrix R is stable if and only if (i) its trace is negative and (ii) its determinant is positive. Remark that (i) is guaranteed by (c) of Assumption 1 and (ii) corresponds to (b) of Assumption 2. Since $r_{jj} < 0$, we conclude that $\partial p_i^* / \partial \gamma_{i0} > 0$. In turn, (c) of Assumption 2 implies that $\partial p_j^* / \partial \gamma_{i0} < 0$. Thus, the comparative statics effects have the ‘normal’ signs. Finally, we note that (c) of Assumption 2 implies that $\gamma_{ij} > -2\alpha_{ij} / \gamma_{jj}$, i.e. it gives a (negative) lowerbound on the size of the coefficient γ_{ij} of the cross-effect of advertising, given the values of α_{ij} and γ_{jj} .

Proceeding, let us turn now to the situation in which the two firms do not make any advertisements, i.e. the advertising expenditures are restricted to $A_1 = A_2 = 0$. One can verify that the price vector $p^0 = (p_1^0, p_2^0)'$, say, in the (interior) Nash equilibrium corresponding to this situation must satisfy the first-order conditions $R_0(p^0 - c) = d$, where matrix R_0 is given by

$$R_0 = \begin{pmatrix} 2\alpha_{11} & \alpha_{12} \\ \alpha_{21} & 2\alpha_{22} \end{pmatrix}. \quad (9)$$

Further, analogously to (7) and (8), the corresponding output level q_i^0 of firm i can be written as

$$q_i^0 = -(p_i^0 - c_i)\alpha_{ii}, \quad (10)$$

and the corresponding profit level $\pi_i^0 = \pi_i(p_i^0, p_j^0, 0, 0)$ of firm i equals

$$\pi_i^0 = -(p_i^0 - c_i)^2 \alpha_{ii}. \quad (11)$$

We remark that the Assumptions 1 and 2 also guarantee that this Nash equilibrium is well defined, i.e. $p_i^0 - c_i > 0$, $q_i^0 > 0$ and $\pi_i^0 > 0$ for $i = 1, 2$. Also notice that all elements of matrix R_0^{-1} in $p^0 - c = R_0^{-1}d$ are negative.

3. The profits in the Nash equilibria

In this section we will compare the size of the profits of firm i in the Nash equilibrium $(p_1^*, p_2^*, A_1^*, A_2^*)$ with advertising and the Nash equilibrium (p_1^0, p_2^0) without advertising, i.e. we compare π_i^* and π_i^0 , $i = 1, 2$.

First, we recall from the previous section that $p^* - c = R^{-1}d$ and $p^0 - c = R_0^{-1}d$. Next, it follows from (8) and (11) that $\pi_i^0 > \pi_i^*$ if and only if $(p_i^0 - c_i)^2 > (1 +$

¹ This assumption is known as the correspondence principle of Samuelson (1947); see Gandolfo (1997, chapter 20) for a comprehensive modern discussion of the principle. For a critical account of imposing an *ad hoc* (stable) dynamic adjustment process upon a static model, see Caputo (1996).

$\frac{\gamma_{ii}^2}{4\alpha_{ii}})(p_i^* - c_i)^2, i = 1, 2$. Now, define matrix $T = (t_{ij})$ as

$$T = R_0^{-1} - SR^{-1}, \quad (12)$$

where matrix $S = (s_{ij})$ equals

$$S = \begin{pmatrix} \sqrt{1 + \frac{\gamma_{11}^2}{4\alpha_{11}}} & 0 \\ 0 & \sqrt{1 + \frac{\gamma_{22}^2}{4\alpha_{22}}} \end{pmatrix}. \quad (13)$$

Notice that as a result of (c) of Assumption 1, the diagonal elements of matrix S satisfy $0 < s_{ii} < 1, i = 1, 2$. Combining results, we obtain that

$$\pi_i^* \geq \pi_i^0 \Leftrightarrow t_{ii}d_i + t_{ij}d_j \leq 0, \quad i, j = 1, 2 \ (j \neq i). \quad (14)$$

Recall that $d_i < 0, i = 1, 2$. Further, recalling that the elements of R_0^{-1} and SR^{-1} are negative, we conclude that the elements of T may be positive, negative or equal to zero. Note that the elements of T do not depend on the parameters $\gamma_{i0}, i = 1, 2$.

For a given value of i , we will analyse now the signs of t_{ii} and t_{ij} as a function of the advertising cross-effect coefficients γ_{ij} and γ_{ji} , i.e. we write $t_{ii} = t_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij} = t_{ij}(\gamma_{ij}, \gamma_{ji})$. Remark that given the values of $\alpha_{11}, \alpha_{22}, \alpha_{12}, \alpha_{21}, \gamma_{11}$ and γ_{22} , the values of γ_{ij} and γ_{ji} are constrained by (b) and (c) of Assumption 2. In particular, γ_{ij} and γ_{ji} must lie in a feasible region Γ_i , say, of the $(\gamma_{ij}, \gamma_{ji})$ -plane, where Γ_i consists of all points $(\gamma_{ij}, \gamma_{ji})$ that simultaneously satisfy the following three requirements: (i) they lie above a lower boundary given by the line $\gamma_{ji} = -2\alpha_{ji}/\gamma_{ii}$, (ii) they lie to the right of a left boundary given by the line $\gamma_{ij} = -2\alpha_{ij}/\gamma_{jj}$, and (iii) they lie below and to the left of an upper-right boundary given by the points $(\gamma_{ij}, \gamma_{ji})$ such that $\det(R) = 0$. The three boundaries themselves are not part of Γ_i . The slope of the upper-right boundary in the point $(\gamma_{ij}, \gamma_{ji})$ is given by

$$\left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|\det(R)=0} = -\frac{\partial \det(R)}{\partial \gamma_{ij}} / \frac{\partial \det(R)}{\partial \gamma_{ji}} = -\frac{\gamma_{jj}(\alpha_{ji} + \frac{1}{2}\gamma_{ji}\gamma_{ii})}{\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})} < 0. \quad (15)$$

Thus, the upper-right boundary is downward sloping and has a strictly convex form towards the point $(-2\alpha_{ij}/\gamma_{jj}, -2\alpha_{ji}/\gamma_{ii})$. Also notice that

$$\lim_{\gamma_{ij} \downarrow -\frac{2\alpha_{ij}}{\gamma_{jj}}} \left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|\det(R)=0} = -\infty, \quad \lim_{\gamma_{ji} \downarrow -\frac{2\alpha_{ji}}{\gamma_{ii}}} \left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|\det(R)=0} = 0. \quad (16)$$

The set of all points $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ such that $t_{ii}(\gamma_{ij}, \gamma_{ji}) = \bar{t}_{ii}$, where \bar{t}_{ii} is some constant, is called a level curve of $t_{ii}(\gamma_{ij}, \gamma_{ji})$. Analogously, $t_{ij}(\gamma_{ij}, \gamma_{ji}) = \bar{t}_{ij}$ defines a level curve of $t_{ij}(\gamma_{ij}, \gamma_{ji})$. The following lemma which is proved in the Appendix, presents an overview with respect to the possible signs of $t_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji})$ on Γ_i .

Lemma 3.1 Take the duopoly model with the Assumptions 1 and 2 and matrix T of (12). Consider level curves of $t_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji})$ in Γ_i , with $i, j = 1, 2$ ($j \neq i$). The following holds:

- (a) Each level curve of $t_{ii}(\gamma_{ij}, \gamma_{ji})$ is downward sloping and strictly convex towards the point $(-2\alpha_{ij}/\gamma_{jj}, -2\alpha_{ji}/\gamma_{ii})$. To the right (resp. left) of the level curve $t_{ii}(\gamma_{ij}, \gamma_{ji}) = \bar{t}_{ii}$ we have $t_{ii}(\gamma_{ij}, \gamma_{ji}) > \bar{t}_{ii}$ (resp. $< \bar{t}_{ii}$). Furthermore

$$\lim_{\gamma_{ij} \downarrow -\frac{2\alpha_{ij}}{\gamma_{jj}}} \left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|t_{ii}=\bar{t}_{ii}} = -\infty, \quad \lim_{\gamma_{ji} \downarrow -\frac{2\alpha_{ji}}{\gamma_{ii}}} \left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|t_{ii}=\bar{t}_{ii}} = 0.$$

- (b) There are two cases with respect to the level curves of $t_{ii}(\gamma_{ij}, \gamma_{ji})$. First, if $4\alpha_{11}\alpha_{22}(1 - s_{ii}) \geq \alpha_{12}\alpha_{21}$, then $t_{ii}(\gamma_{ij}, \gamma_{ji}) > 0$ everywhere on Γ_i . Second, if $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < \alpha_{12}\alpha_{21}$, then Γ_i contains a level curve $t_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$. As a result, in this case $t_{ii}(\gamma_{ij}, \gamma_{ji}) \geq 0$ if and only if $\gamma_{ij} \geq \hat{\gamma}_{ij}(\gamma_{ji})$, where given the value of γ_{ji} , $\hat{\gamma}_{ij}(\gamma_{ji})$ is the unique solution of $t_{ii}(\hat{\gamma}_{ij}(\gamma_{ji}), \gamma_{ji}) = 0$. Further, $\hat{\gamma}_{ij}(\gamma_{ji}) < 0$ for all $\gamma_{ji} \geq 0$.
- (c) Each level curve of $t_{ij}(\gamma_{ij}, \gamma_{ji})$ is downward sloping and strictly convex towards the point $(-2\alpha_{ij}/\gamma_{jj}, -2\alpha_{ji}/\gamma_{ii})$. To the right (resp. left) of the level curve $t_{ij}(\gamma_{ij}, \gamma_{ji}) = \bar{t}_{ij}$ we have $t_{ij}(\gamma_{ij}, \gamma_{ji}) > \bar{t}_{ij}$ (resp. $< \bar{t}_{ij}$). Furthermore

$$\lim_{\gamma_{ij} \downarrow -\frac{2\alpha_{ij}}{\gamma_{jj}}} \left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|t_{ij}=\bar{t}_{ij}} = -\infty.$$

- (d) Γ_i contains a level curve $t_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$. As a result, $t_{ij}(\gamma_{ij}, \gamma_{ji}) \geq 0$ if and only if $\gamma_{ij} \geq \tilde{\gamma}_{ij}(\gamma_{ji})$, where given the value of γ_{ji} , $\tilde{\gamma}_{ij}(\gamma_{ji})$ is the unique solution of $t_{ij}(\tilde{\gamma}_{ij}(\gamma_{ji}), \gamma_{ji}) = 0$. Further, $\tilde{\gamma}_{ij}(\gamma_{ji}) < 0$ for all $\gamma_{ji} \geq 0$. Finally, there exists on the lower boundary of Γ_i a point $(\gamma_{ij}^0, -2\alpha_{ji}/\gamma_{ii})$ with $\gamma_{ij}^0 > -2\alpha_{ij}/\gamma_{jj}$ such that $t_{ij}(\gamma_{ij}^0, -2\alpha_{ji}/\gamma_{ii}) = 0$. There holds $\gamma_{ij}^0 \leq 0$ if and only if $4\alpha_{11}\alpha_{22}(1 - s_{ii}s_{jj}^2) \geq \alpha_{12}\alpha_{21}$.
- (e) Level curves of $t_{ij}(\gamma_{ij}, \gamma_{ji})$ are steeper than level curves of $t_{ii}(\gamma_{ij}, \gamma_{ji})$, i.e. in each $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ the slope of the corresponding level curve of $t_{ij}(\gamma_{ij}, \gamma_{ji})$ is in absolute value greater than the slope of the corresponding level curve of $t_{ii}(\gamma_{ij}, \gamma_{ji})$.
- (f) If $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < \alpha_{12}\alpha_{21}$, then the level curves $t_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$ and $t_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$ have a unique point of intersection $(\gamma_{ij}^s, \gamma_{ji}^s) \in \Gamma_i$, say, with $\gamma_{ij}^s < 0$ and $\gamma_{ji}^s < 0$.

In Figure 3.1 we illustrate Lemma 3.1 for a situation in which $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < \alpha_{12}\alpha_{21}$ and $\gamma_{ij}^0 > 0$. We further make the following remark with respect to Lemma 3.1.

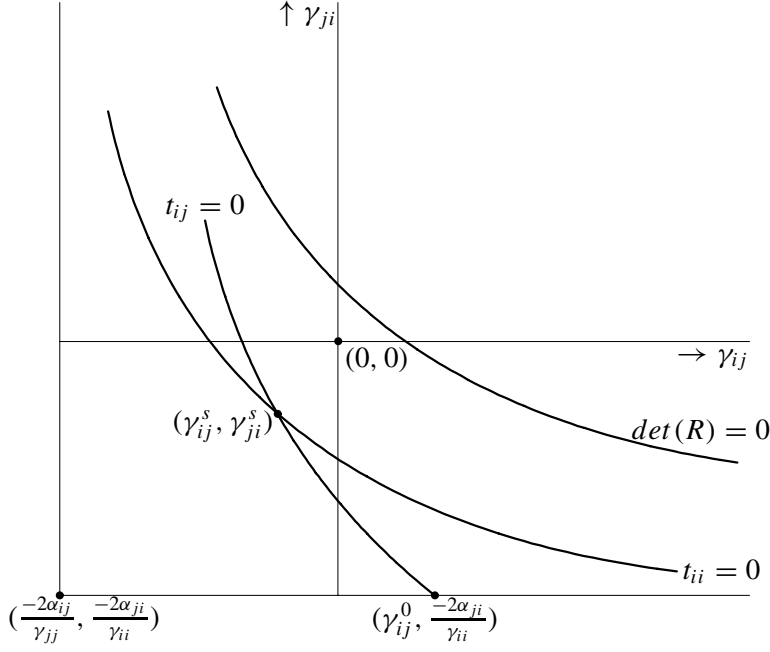


Figure 3.1: level curves $t_{ii} = 0$ and $t_{ij} = 0$.

According to (b) of Assumption 1 we have $\alpha_{11}\alpha_{22} > \alpha_{12}\alpha_{21}$, and thus certainly $4\alpha_{11}\alpha_{22} > \alpha_{12}\alpha_{21}$. Using (13), this shows that the case $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < \alpha_{12}\alpha_{21}$, mentioned in (b) of Lemma 3.1, can occur only if, given the values of α_{11} , α_{22} , α_{12} and α_{21} , the size of firm i 's 'own-advertising' coefficient γ_{ii} is 'small enough'. Similarly, the case $4\alpha_{11}\alpha_{22}(1 - s_{ii}s_{jj}^2) < \alpha_{12}\alpha_{21}$, mentioned in (d) of Lemma 3.1, can occur only if, given the values of α_{11} , α_{22} , α_{12} , α_{21} and γ_{jj} , the size of γ_{ii} is 'extremely small'. Note that always $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < 4\alpha_{11}\alpha_{22}(1 - s_{ii}s_{jj}^2)$. We will say below that the effect of advertising of firm i on its own demand is 'relatively large' if $4\alpha_{11}\alpha_{22}(1 - s_{ii}) \geq \alpha_{12}\alpha_{21}$. The effect is 'relatively small' if $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < \alpha_{12}\alpha_{21}$.

Using Lemma 3.1 we are able to present the following proposition which characterizes in a precise way the circumstances in which the profit π_i^* of firm i in the Nash equilibrium with advertising is greater than (resp. smaller than, or equal to) the profit π_i^0 in the Nash equilibrium without advertising.

Proposition 3.1 *Take the duopoly model with the Assumptions 1 and 2 and matrix T defined in (12). For $i, j = 1, 2$ ($j \neq i$), let $\hat{\gamma}_{ij}(\gamma_{ji})$, $\tilde{\gamma}_{ij}(\gamma_{ji})$ and $(\gamma_{ij}^s, \gamma_{ji}^s)$ be as*

defined in, respectively, (b), (d) and (f) of Lemma 3.1. Finally, define

$$\hat{\gamma}_{i0} = (t_{ij}/t_{ii})d_j - \alpha_{ii}c_i - \alpha_{ij}c_j.$$

Then the following cases can be distinguished for $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$:

Case (i): Let $4\alpha_{11}\alpha_{22}(1 - s_{ii}) \geq \alpha_{12}\alpha_{21}$. Then we have for each value of γ_{ji} :

- (a) If $\gamma_{ij} < \tilde{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* \geq \pi_i^0$ if and only if $\gamma_{i0} \geq \hat{\gamma}_{i0}$.
- (b) If $\gamma_{ij} \geq \tilde{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* > \pi_i^0$.

Case (ii): Let $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < \alpha_{12}\alpha_{21}$.

- subcase (iia): Let $\gamma_{ji} < \gamma_{ji}^s$. Then $\tilde{\gamma}_{ij}(\gamma_{ji}) < \hat{\gamma}_{ij}(\gamma_{ji})$, and we have:

- (c) If $\gamma_{ij} \leq \tilde{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* < \pi_i^0$.
- (d) If $\tilde{\gamma}_{ij}(\gamma_{ji}) < \gamma_{ij} < \hat{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* \geq \pi_i^0$ if and only if $\gamma_{i0} \leq \hat{\gamma}_{i0}$.
- (e) If $\hat{\gamma}_{ij}(\gamma_{ji}) \leq \gamma_{ij}$, then $\pi_i^* > \pi_i^0$.

- subcase (iib): Let $\gamma_{ji} = \gamma_{ji}^s$. Then $\tilde{\gamma}_{ij}(\gamma_{ji}) = \hat{\gamma}_{ij}(\gamma_{ji})$, and we have:

- (f) If $\gamma_{ij} < \tilde{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* < \pi_i^0$.
- (g) If $\gamma_{ij} = \tilde{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* = \pi_i^0$.
- (h) If $\tilde{\gamma}_{ij}(\gamma_{ji}) < \gamma_{ij}$, then $\pi_i^* > \pi_i^0$.

- subcase (iic): Let $\gamma_{ji} > \gamma_{ji}^s$. Then $\hat{\gamma}_{ij}(\gamma_{ji}) < \tilde{\gamma}_{ij}(\gamma_{ji})$, and we have:

- (i) If $\gamma_{ij} \leq \hat{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* < \pi_i^0$.
- (j) If $\hat{\gamma}_{ij}(\gamma_{ji}) < \gamma_{ij} < \tilde{\gamma}_{ij}(\gamma_{ji})$, then $\pi_i^* \geq \pi_i^0$ if and only if $\gamma_{i0} \geq \hat{\gamma}_{i0}$.
- (k) If $\tilde{\gamma}_{ij}(\gamma_{ji}) \leq \gamma_{ij}$, then $\pi_i^* > \pi_i^0$.

PROOF. We only give the proof of parts (a) and (b). All other parts can be proved similarly. First, recall (14) and the fact that $d_i < 0$, $i = 1, 2$. Next, let $4\alpha_{11}\alpha_{22}(1 - s_{ii}) \geq \alpha_{12}\alpha_{21}$. In that case we know from (b) of Lemma 3.1 that always $t_{ii}(\gamma_{ij}, \gamma_{ji}) > 0$. Furthermore, in case $\gamma_{ij} \geq \tilde{\gamma}_{ij}(\gamma_{ji})$, it follows that $t_{ij}(\gamma_{ij}, \gamma_{ji}) \geq 0$, and we conclude from (14) that $\pi_i^* > \pi_i^0$, which proves (b) of the proposition. On the other side, if $\gamma_{ij} < \tilde{\gamma}_{ij}(\gamma_{ji})$, then $t_{ij}(\gamma_{ij}, \gamma_{ji}) < 0$, and we conclude from (14) that $\pi_i^* \geq \pi_i^0$ if and only if $\gamma_{i0} \geq \hat{\gamma}_{i0}$, which establishes (a) of the proposition. \square

Discussing Proposition 3.1 it is useful to give corollaries for the two interesting symmetric cases in which advertising of both firms has either a stimulating or an adverse cross-effect on the demand of its rival. We can derive in a similar way a corollary for the case in which advertising of firm j , say, has a stimulating cross-effect on firm i , whereas advertising of firm i has an adverse cross-effect on firm j . Details of this asymmetric case are left to the reader. First, we see that the profits are greatest in

the Nash equilibrium with advertising if advertising of both firms has a stimulating cross-effect, i.e.

Corollary 3.1 *Take the duopoly model with the Assumptions 1 and 2. Consider the case in which advertising of both firms has a stimulating cross-effect on the demand of the other firm. Then $\pi_i^* > \pi_i^0$, for $i = 1, 2$.*

PROOF. We know from (b) and (d) of Lemma 3.1 that $\tilde{\gamma}_{ij}(\gamma_{ji}) < 0$ and $\hat{\gamma}_{ij}(\gamma_{ji}) < 0$ for all $\gamma_{ji} > 0$. The proof then follows from (b), (e), (h) and (k) of Proposition 3.1. \square

Next, we remark that the terminology used in Corollary 3.2 corresponds in the obvious way to the relevant cases distinguished in Proposition 3.1. In particular, we mention that parts (1) and (2) of Corollary 3.2 correspond to, respectively, (a) and (b) of Proposition 3.1. Parts (3) to (8) correspond to, respectively, (c), (d), (e), (i), (j) and (k) of the proposition.

Corollary 3.2 *Take the duopoly model with the Assumptions 1 and 2. Consider the case in which advertising of both firms has an adverse cross-effect on the demand of the other firm. We then can identify the following typical situations regarding π_i^* and π_i^0 , with $i, j = 1, 2$ ($j \neq i$):*

- *Suppose that the effect of the advertising of firm i on its own demand is relatively large. Then:*

- (1) *If advertising of firm j has a relatively strongly adverse cross-effect on firm i 's demand, then π_i^* is greater than π_i^0 if and only if the autonomous demand of firm i is relatively large.*
- (2) *If advertising of firm j has a relatively weakly adverse cross-effect on firm i 's demand, then π_i^* is greater than π_i^0 .*

- *Suppose that the effect of the advertising of firm i on its own demand is relatively small. Then:*

- (3) *If advertising of firm i has a relatively strongly adverse cross-effect on firm j 's demand and advertising of firm j has a relatively strongly adverse cross-effect on firm i 's demand, then π_i^* is smaller than π_i^0 .*
- (4) *If advertising of firm i has a relatively strongly adverse cross-effect on firm j 's demand and advertising of firm j has a relatively moderately adverse cross-effect on firm i 's demand, then π_i^* is greater than π_i^0 if and only if the autonomous demand of firm i is relatively small.*
- (5) *If advertising of firm i has a relatively strongly adverse cross-effect on firm j 's demand and advertising of firm j has a relatively weakly adverse cross-effect*

- on firm i 's demand, then π_i^* is greater than π_i^0 .
- (6) If advertising of firm i has a relatively weakly adverse cross-effect on firm j 's demand and advertising of firm j has a relatively strongly adverse cross-effect on firm i 's demand, then π_i^* is smaller than π_i^0 .
 - (7) If advertising of firm i has a relatively weakly adverse cross-effect on firm j 's demand and advertising of firm j has a relatively moderately adverse cross-effect on firm i 's demand, then π_i^* is greater than π_i^0 if and only if the autonomous demand of firm i is relatively large.
 - (8) If advertising of firm i has a relatively weakly adverse cross-effect on firm j 's demand and advertising of firm j has a relatively weakly adverse effect on firm i 's demand, then π_i^* is greater than π_i^0 .

Clearly, Corollary 3.2 lends an intuitive explanation of the determinants of the relative size of π_i^* and π_i^0 . For instance, comparing parts (2) and (1), we see that if the advertising of firm j becomes more adverse with respect to firm i 's demand, then an additional condition must hold in order to have that π_i^* is greater than π_i^0 , i.e. the autonomous demand of firm i must be relatively large ($\gamma_{i0} > \hat{\gamma}_{i0}$). In a similar way, we can point out the effect if the advertising of firm j becomes more adverse with respect to firm i 's demand, by comparing the parts (5), (4) and (3), or the parts (8), (7) and (6).

Viewed from a different angle, we can say that part (2) of Corollary 3.2 describes the typical 'extreme' case in which advertising is mostly advantageous for firm i , i.e. the (positive) effect of advertising of firm i on its own demand is relatively large and, in addition, advertising of firm j has only a relatively weakly adverse cross-effect on firm i 's demand. So, it is clear that $\pi_i^* > \pi_i^0$ in this case. On the other hand, parts (3) and (6) describe the opposite 'extreme' cases in which advertising is mostly disadvantageous for firm i , because the (positive) effect of advertising of firm i on its own demand is now only relatively small and, moreover, advertising of firm j has a relatively strongly adverse cross-effect on firm i 's demand. Again, it is clear that now $\pi_i^* < \pi_i^0$. Note that the parts (2), (3) and (6) do not depend on the size of the autonomous demand of firm i . Further, the other parts of the corollary capture the 'intermediate' situations which lie between these 'extreme' ones.

We further notice the difference between the parts (4) and (7) of Corollary 3.2. Apparently, there is a trade-off with respect to the determinants of the case $\pi_i^* > \pi_i^0$: i.e. *ceteris paribus* we have $\pi_i^* > \pi_i^0$ if either advertising of firm i has a relatively *strongly* adverse cross-effect on firm j 's demand and the autonomous demand of firm i is relatively *small* (part (4)), or advertising of firm i has a relatively *weakly* adverse cross-effect on firm j 's demand and the autonomous demand of firm i is relatively

large (part (7)).

Concluding this section, we remark that Proposition 3.1 gives us a complete characterization of all possible situations regarding the size of the profits of firm i in the two Nash equilibria. Broadly speaking, we can say that the comparison of the profit levels is relatively simple if the effect of advertising of firm i on its own demand is relatively large, see case (i) of Proposition 3.1. Namely, in that case only two different situations are possible, i.e. parts (a) and (b) of the proposition. On the other side, if the effect of advertising of firm i on its own demand is relatively small (see case (ii) of Proposition 3.1), then the comparison of the profits is much more tedious, as is reflected in the larger number of possible situations, i.e. parts (c) up to and including (k) of the proposition. From a theoretical point of view, we have to take into account the possible occurrence of all different situations. Clearly, empirical work is needed in order to assess which one is relevant in a specific practical application.

4. The outputs and prices in the Nash equilibria

In this section we briefly discuss the comparison of the outputs and prices of firm i in the Nash equilibria with and without advertising, i.e. we compare q_i^* and p_i^* with q_i^0 and p_i^0 , for $i = 1, 2$. Recall $p^* - c = R^{-1}d$, $p^0 - c = R_0^{-1}d$, and (7) and (10). Defining matrix $U = (u_{ij})$ as

$$U = R_0^{-1} - R^{-1} \quad (17)$$

we obtain

$$q_i^* \geq q_i^0 \Leftrightarrow p_i^* \geq p_i^0 \Leftrightarrow u_{ii}d_i + u_{ij}d_j \leq 0, \quad i, j = 1, 2 \ (j \neq i). \quad (18)$$

Clearly, (18) is the counterpart of (14). The only difference between the matrices T and U is that in the definition of matrix U the matrix S is replaced by an identity matrix. In fact, the comparison of the outputs and prices in the two Nash equilibria can be carried out along exactly the same lines as the comparison of the profits in the previous section.

In particular, let us define in Γ_i level curves $u_{ii}(\gamma_{ij}, \gamma_{ji}) = \bar{u}_{ii}$ and $u_{ij}(\gamma_{ij}, \gamma_{ji}) = \bar{u}_{ij}$. It can be verified that in each point $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ we have $t_{ii}(\gamma_{ij}, \gamma_{ji}) < u_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji}) < u_{ij}(\gamma_{ij}, \gamma_{ji})$. Furthermore, in each $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ the slopes of the corresponding level curves of $u_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ii}(\gamma_{ij}, \gamma_{ji})$ are identical. The same applies to the slopes of the level curves of $u_{ij}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji})$. Using this, we easily obtain the following counterpart of Lemma 3.1.

Lemma 4.1 *Take the duopoly model with the Assumptions 1 and 2, matrix T of (12) and matrix U of (17). Consider level curves of $u_{ii}(\gamma_{ij}, \gamma_{ji})$ and $u_{ij}(\gamma_{ij}, \gamma_{ji})$*

in Γ_i , with $i, j = 1, 2$ ($j \neq i$). The following holds:

- (a) Each level curve $u_{ii}(\gamma_{ij}, \gamma_{ji}) = \bar{u}_{ii}$ coincides with some level curve $t_{ii}(\gamma_{ij}, \gamma_{ji}) = \bar{t}_{ii}$, where $\bar{t}_{ii} < \bar{u}_{ii}$. To the right (resp. left) of the level curve $u_{ii}(\gamma_{ij}, \gamma_{ji}) = \bar{u}_{ii}$ we have $u_{ii}(\gamma_{ij}, \gamma_{ji}) > \bar{u}_{ii}$ (resp. $< \bar{u}_{ii}$).
- (b) There are two cases with respect to the level curves of $u_{ii}(\gamma_{ij}, \gamma_{ji})$. First, if $4\alpha_{11}\alpha_{22}(1 - s_{ii}^2) \geq \alpha_{12}\alpha_{21}$, then $u_{ii}(\gamma_{ij}, \gamma_{ji}) > 0$ everywhere on Γ_i . Second, if $4\alpha_{11}\alpha_{22}(1 - s_{ii}^2) < \alpha_{12}\alpha_{21}$, then Γ_i contains a level curve $u_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$. As a result, in this case $u_{ii}(\gamma_{ij}, \gamma_{ji}) \geq 0$ if and only if $\gamma_{ij} \geq \bar{\gamma}_{ij}(\gamma_{ji})$, where given the value of γ_{ji} , $\bar{\gamma}_{ij}(\gamma_{ji})$ is the unique solution of $u_{ii}(\bar{\gamma}_{ij}(\gamma_{ji}), \gamma_{ji}) = 0$. Further, $\bar{\gamma}_{ij}(\gamma_{ji}) < 0$ for all $\gamma_{ji} \geq 0$.
- (c) Each level curve $u_{ij}(\gamma_{ij}, \gamma_{ji}) = \bar{u}_{ij}$ coincides with some level curve $t_{ij}(\gamma_{ij}, \gamma_{ji}) = \bar{t}_{ij}$, where $\bar{t}_{ij} < \bar{u}_{ij}$. To the right (resp. left) of the level curve $u_{ij}(\gamma_{ij}, \gamma_{ji}) = \bar{u}_{ij}$ we have $u_{ij}(\gamma_{ij}, \gamma_{ji}) > \bar{u}_{ij}$ (resp. $< \bar{u}_{ij}$).
- (d) Γ_i contains a level curve $u_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$. As a result, $u_{ij}(\gamma_{ij}, \gamma_{ji}) \geq 0$ if and only if $\gamma_{ij} \geq \bar{\gamma}_{ij}(\gamma_{ji})$, where given the value of γ_{ji} , $\bar{\gamma}_{ij}(\gamma_{ji})$ is the unique solution of $u_{ij}(\bar{\gamma}_{ij}(\gamma_{ji}), \gamma_{ji}) = 0$. Further, $\bar{\gamma}_{ij}(\gamma_{ji}) < 0$ for all $\gamma_{ji} \geq 0$. Finally, there exists on the lower boundary of Γ_i a point $(\gamma_{ij}^1, -2\alpha_{ji}/\gamma_{ii})$ with $\gamma_{ij}^1 > -2\alpha_{ij}/\gamma_{jj}$ such that $u_{ij}(\gamma_{ij}^1, -2\alpha_{ji}/\gamma_{ii}) = 0$. There holds $\gamma_{ij}^1 \leq 0$ if and only if $4\alpha_{11}\alpha_{22}(1 - s_{ii}^2 s_{jj}^2) \geq \alpha_{12}\alpha_{21}$.
- (e) If $4\alpha_{11}\alpha_{22}(1 - s_{ii}^2) < \alpha_{12}\alpha_{21}$, then the level curves $u_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$ and $u_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$ have a unique point of intersection $(\gamma_{ij}^{ss}, \gamma_{ji}^{ss}) \in \Gamma_i$, say, with $\gamma_{ij}^{ss} < 0$ and $\gamma_{ji}^{ss} < 0$.

We make now three observations. First, part (a) of Lemma 4.1 implies that a level curve of $u_{ii}(\gamma_{ij}, \gamma_{ji})$ which corresponds to a certain constant value is located below and to the left of the level curve of $t_{ii}(\gamma_{ij}, \gamma_{ji})$ pertaining to the same constant value. Part (c) means that the same applies to the level curves of $u_{ij}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji})$. Hence, the level curve $u_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$ (in case it lies within Γ_i) and the level curve $u_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$ are located below and to the left of the level curves $t_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$ and $t_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$, respectively. As a result, for all γ_{ji} there holds $\bar{\gamma}_{ij}(\gamma_{ji}) < \hat{\gamma}_{ij}(\gamma_{ji})$ and $\bar{\gamma}_{ij}(\gamma_{ji}) < \tilde{\gamma}_{ij}(\gamma_{ji})$.

Second, using Lemma 4.1, we can give in the obvious way results completely similar to Proposition 3.1 and its corollaries with respect to the comparison of q_i^* versus q_i^0 , and p_i^* versus p_i^0 . For instance, as the counterpart of (b) of Proposition 3.1 we have the following. Let $4\alpha_{11}\alpha_{22}(1 - s_{ii}^2) \geq \alpha_{12}\alpha_{21}$. Then we have for each value of γ_{ji} : if $\gamma_{ij} \geq \bar{\gamma}_{ij}(\gamma_{ji})$, then $q_i^* > q_i^0$ and $p_i^* > p_i^0$.

Third, using the first two observations, and recalling our discussion of advertising bans in Section 1, we observe that there are situations with $-2\alpha_{ij}/\gamma_{jj} < \gamma_{ij} < 0$ and $-2\alpha_{ji}/\gamma_{ii} < \gamma_{ji} < 0$ such that $q_i^* > q_i^0$ and $p_i^* > p_i^0$, but $\pi_i^* < \pi_i^0$. In these situations the output and price of firm i are smallest in the Nash equilibrium without advertising. However, the profits are greatest in this Nash equilibrium, i.e. it is in the interest of firm i if the firms (have to) stop with advertising. The reason is that the reduction in the revenues of firm i is more than compensated by the disappearance of its advertising costs. To illustrate the occurrence of such situations, take the case where $4\alpha_{11}\alpha_{22}(1 - s_{ii}) \geq \alpha_{12}\alpha_{21}$, i.e. the effect of advertising of firm i on its own demand is relatively large. Corresponding to this case, we can draw Figure 4.1 (note that in this case $t_{ii}(\gamma_{ij}, \gamma_{ji}) > 0$ and $u_{ii}(\gamma_{ij}, \gamma_{ji}) > 0$ everywhere in Γ_i). Consider now a point in the interior of the shaded region N . We then have $\gamma_{ij} \geq \bar{\gamma}_{ij}(\gamma_{ji})$, and thus $q_i^* > q_i^0$ and $p_i^* > p_i^0$ (cf. our second observation above). Further, we also have $\gamma_{ij} < \tilde{\gamma}_{ij}(\gamma_{ji})$, which implies that $\pi_i^* < \pi_i^0$ if and only if $\gamma_{i0} < \hat{\gamma}_{i0}$, i.e. if and only if the autonomous demand of firm i is relatively small (cf. part (a) of Proposition 3.1). Intuitively speaking, the condition that the autonomous demand of firm i must be relatively small can be understood by noting that, *ceteris paribus*, in that case a decrease in the price of firm i from p_i^* to p_i^0 leads to a relatively small reduction in the revenues of firm i .

5. Conclusion

This paper has analysed the impact of advertising in a duopoly model by comparing the two situations in which the firms either simultaneously compete in prices as well as in advertising, or do not advertise at all and only compete in prices. We presented a proposition that characterizes in terms of a small set of factors the relative size of the profit of firm $i = 1, 2$ in the two corresponding Nash equilibria. The relevant factors are (a) the size of the (positive) effect of advertising of firm i on its own demand, (b) the size and nature (stimulating or adverse) of the cross-effect of the advertising of each firm on the demand of the other firm, and (c) the size of the autonomous demand of firm i .

Interpreting, we focused on the two symmetric cases in which advertising of both firms has either a stimulating or an adverse cross-adverse effect on the demand of its rival. For the case with stimulating cross-effects, the profit of firm i is highest in the Nash equilibrium with advertising. For the case with adverse cross-effects, we identified two opposite ‘extreme’ situations. In the first one, the effect of advertising of firm i on its own demand is relatively large and, in addition, advertising of firm $j \neq i$ has a relatively weakly adverse effect on firm i ’s demand. In that case the profit

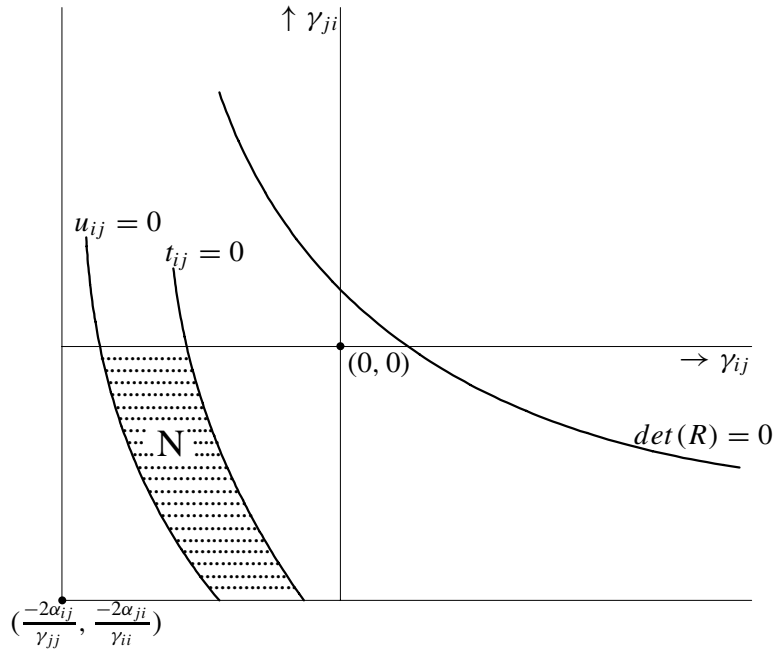


Figure 4.1: relevant region for the case $q_i^* > q_i^0$, $p_i^* > p_i^0$, and $\pi_i^* < \pi_i^0$.

of firm i is greatest in the Nash equilibrium with advertising. In the second one, the effect of advertising of firm i on its own demand is relatively small and, moreover, advertising of firm $j \neq i$ has a relatively strongly adverse effect on firm i 's demand. Then the profit of firm i is greatest in the Nash equilibrium without advertising. Our results further show that in the situations which lie between these 'extreme' ones, the relative size of the profit of firm i depends on a combination of the factors (a), (b) and/or (c) mentioned above. We have seen that qualitatively identical conclusions hold with respect to the relative size of the outputs and prices of each firm in the two Nash equilibria. Concluding, we remark that our theoretical analysis has provided us with a full characterization of all possible situations. Clearly, empirical work is needed in order to assess which situation is relevant in specific practical applications.

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Appendix A:

In this appendix we provide the proof of Lemma 3.1 and give some details with respect to the elements of the matrix U of (17).

Straightforward manipulations show that the elements of matrix T of (12) are given by

$$\begin{aligned} t_{ii} &= \frac{2\alpha_{jj}}{\det(R_0)} - \left[\frac{s_{ii}(2\alpha_{jj} + \frac{1}{2}\gamma_{jj}^2)}{\det(R)} \right] \\ &= \frac{2\alpha_{jj}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} - \\ &\quad \left[\frac{s_{ii}(2\alpha_{jj} + \frac{1}{2}\gamma_{jj}^2)}{4\alpha_{11}\alpha_{22}s_{11}^2s_{22}^2 - (\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{21}\gamma_{11})} \right] \end{aligned} \quad (\text{A.1})$$

and

$$\begin{aligned} t_{ij} &= -\frac{\alpha_{ij}}{\det(R_0)} + \left[\frac{s_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})}{\det(R)} \right] \\ &= -\frac{\alpha_{ij}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} + \\ &\quad \left[\frac{s_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})}{4\alpha_{11}\alpha_{22}s_{11}^2s_{22}^2 - (\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{21}\gamma_{11})} \right], \end{aligned} \quad (\text{A.2})$$

where $i, j = 1, 2$ ($j \neq i$).

• Proof of Lemma 3.1:

- (a) Observe that in all points of Γ_i there holds

$$\begin{aligned} \frac{\partial t_{ii}}{\partial \gamma_{ij}} &= \frac{-s_{ii}(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)\gamma_{jj}(\alpha_{ji} + \frac{1}{2}\gamma_{ji}\gamma_{ii})}{(\det(R))^2} > 0 \\ \frac{\partial t_{ii}}{\partial \gamma_{ji}} &= \frac{-s_{ii}(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})}{(\det(R))^2} > 0. \end{aligned}$$

Using this, the slope of the level curve of $t_{ii}(\gamma_{ij}, \gamma_{ji})$ in the point $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ is given by

$$\left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|_{t_{ii}=\bar{t}_{ii}}} = -\frac{\partial t_{ii}}{\partial \gamma_{ij}} / \frac{\partial t_{ii}}{\partial \gamma_{ji}} = -\frac{\gamma_{jj}(\alpha_{ji} + \frac{1}{2}\gamma_{ji}\gamma_{ii})}{\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})} < 0. \quad (\text{A.3})$$

All statements of (a) directly follow from these three equations.

- (b) Examining points on the left boundary and lower boundary of Γ_i , we obtain for all γ_{ji} and γ_{ij} that

$$\begin{aligned} t_{ii}\left(-\frac{2\alpha_{ij}}{\gamma_{jj}}, \gamma_{ji}\right) &= t_{ii}\left(\gamma_{ij}, -\frac{2\alpha_{ji}}{\gamma_{ii}}\right) \\ &= \frac{2\alpha_{jj}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} - \left[\frac{s_{ii}(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)}{2\alpha_{11}\alpha_{22}s_{11}^2s_{22}^2} \right] \\ &\equiv \bar{t}_{ii}. \end{aligned}$$

Remark that \bar{t}_{ii} is a constant which is independent of γ_{ij} and γ_{ji} . It can be verified that $\bar{t}_{ii} \geq 0$ if and only if $4\alpha_{11}\alpha_{22}(1 - s_{ii}) \geq \alpha_{12}\alpha_{21}$. Next, we conclude that if $\bar{t}_{ii} \geq 0$, then $t_{ii}(\gamma_{ij}, \gamma_{ji}) > 0$ everywhere on Γ_i . On the contrary, if $\bar{t}_{ii} < 0$, then a level curve $t_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$ must lie in Γ_i .

The statement that $t_{ii}(\gamma_{ij}, \gamma_{ji}) \geq 0$ if and only if $\gamma_{ij} \geq \hat{\gamma}_{ij}(\gamma_{ji})$, where $\hat{\gamma}_{ij}(\gamma_{ji})$ is the unique solution of $t_{ii}(\hat{\gamma}_{ij}(\gamma_{ji}), \gamma_{ji}) = 0$, follows from (a) of the lemma. Finally, it can be verified that there exists a value γ'_{ji} with $-2\alpha_{ji}/\gamma_{ii} < \gamma'_{ji} < 0$, such that $\hat{\gamma}_{ij}(\gamma_{ji}) \leq 0$ if and only if $\gamma_{ji} \geq \gamma'_{ji}$. This implies that $\hat{\gamma}_{ij}(\gamma_{ji}) < 0$ for all $\gamma_{ji} \geq 0$.

- (c) Observe that in all points of Γ_i we have

$$\begin{aligned} \frac{\partial t_{ij}}{\partial \gamma_{ij}} &= \frac{2s_{ii}\gamma_{jj}(\alpha_{ii} + \frac{1}{4}\gamma_{ii}^2)(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)}{(\det(R))^2} > 0 \\ \frac{\partial t_{ij}}{\partial \gamma_{ji}} &= \frac{\frac{1}{2}s_{ii}\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})^2}{(\det(R))^2} > 0. \end{aligned}$$

As a result, the slope of the level curve of $t_{ij}(\gamma_{ij}, \gamma_{ji})$ in the point $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ is given by

$$\begin{aligned} \left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|_{t_{ij}=\bar{t}_{ij}}} &= -\frac{\partial t_{ij}}{\partial \gamma_{ij}} / \frac{\partial t_{ij}}{\partial \gamma_{ji}} \\ &= -\frac{4\gamma_{jj}(\alpha_{ii} + \frac{1}{4}\gamma_{ii}^2)(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)}{\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})^2} < 0. \end{aligned} \tag{A.4}$$

The statements of (c) all follow directly from these three equations.

- (d) Observe that in all points on the left boundary of Γ_i , i.e. for all γ_{ji} , there holds

$$t_{ij}(-\frac{2\alpha_{ij}}{\gamma_{jj}}, \gamma_{ji}) = -\frac{\alpha_{ij}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} < 0,$$

whereas for all points on the lower boundary of Γ_i , i.e. for all γ_{ij} , we have

$$t_{ij}(\gamma_{ij}, -\frac{2\alpha_{ji}}{\gamma_{ii}}) = \frac{-\alpha_{ij}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} + \frac{s_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})}{4\alpha_{11}\alpha_{22}s_{11}^2s_{22}^2}. \quad (\text{A.5})$$

It can be verified from (A.5) that there exists a value $\gamma_{ij}^0 > -2\alpha_{ij}/\gamma_{jj}$ such that $t_{ij}(\gamma_{ij}^0, -2\alpha_{ji}/\gamma_{ii}) = 0$. This implies that Γ_i contains a level curve $t_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$ which, if extended on the lower boundary of Γ_i , passes through the point $(\gamma_{ij}^0, -2\alpha_{ji}/\gamma_{ii})$. Substituting γ_{ij}^0 in (A.5), we can derive that $\gamma_{ij}^0 \leq 0$ if and only if $4\alpha_{11}\alpha_{22}(1 - s_{ii}s_{jj}^2) \geq \alpha_{12}\alpha_{21}$.

The statement that $t_{ij}(\gamma_{ij}, \gamma_{ji}) \geq 0$ if and only if $\gamma_{ij} \geq \tilde{\gamma}_{ij}(\gamma_{ji})$, where $\tilde{\gamma}_{ij}(\gamma_{ji})$ is the unique solution of $t_{ij}(\tilde{\gamma}_{ij}(\gamma_{ji}), \gamma_{ji}) = 0$, follows from (a) of the lemma. Further, two possible cases might occur. First, if $\gamma_{ij}^0 \leq 0$, then $\tilde{\gamma}_{ij}(\gamma_{ji}) < 0$ for all feasible values of γ_{ji} . Second, if $\gamma_{ij}^0 > 0$, then there exists a value γ_{ji}'' with $-2\alpha_{ji}/\gamma_{ii} < \gamma_{ji}'' < 0$, such that $\tilde{\gamma}_{ij}(\gamma_{ji}) \leq 0$ if and only if $\gamma_{ji} \geq \gamma_{ji}''$. In both cases it follows that $\tilde{\gamma}_{ij}(\gamma_{ji}) < 0$ for all $\gamma_{ji} \geq 0$.

- (e) Using (A.3), (A.4) and (b) of Assumption 2 we obtain that

$$\left| \frac{\left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|t_{ij}=\tilde{t}_{ij}}}{\left(\frac{d\gamma_{ji}}{d\gamma_{ij}} \right)_{|t_{ii}=\tilde{t}_{ii}}} \right| = \frac{4\alpha_{11}\alpha_{22}s_{11}^2s_{22}^2}{(\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{21}\gamma_{11})} > 1.$$

This shows that the level curves of $t_{ij}(\gamma_{ij}, \gamma_{ji})$ are steeper than the level curves of $t_{ii}(\gamma_{ij}, \gamma_{ji})$.

- (f) It follows from (b) and (e) of this lemma that in case $4\alpha_{11}\alpha_{22}(1 - s_{ii}) < \alpha_{12}\alpha_{21}$, there exists in Γ_i a unique point of intersection $(\gamma_{ij}^s, \gamma_{ji}^s)$, say, of the level curves $t_{ii}(\gamma_{ij}, \gamma_{ji}) = 0$ and $t_{ij}(\gamma_{ij}, \gamma_{ji}) = 0$. Substituting γ_{ij}^s and γ_{ji}^s in (A.1) and (A.2) and combining both resulting equations, we end up with $\gamma_{ij}^s = \gamma_{jj}\alpha_{ij}/(2\alpha_{jj}) < 0$. If we substitute this expression for γ_{ij}^s again in $t_{ii}(\gamma_{ij}^s, \gamma_{ji}^s) = 0$, we obtain after rearranging that

$$4\alpha_{11}\alpha_{22}s_{ii}^2s_{jj}^2 - (\alpha_{ij} + \frac{1}{2}(\frac{\gamma_{jj}\alpha_{ij}}{2\alpha_{jj}})\gamma_{jj})(\alpha_{ji} + \frac{1}{2}\gamma_{ji}^s\gamma_{ii}) = s_{ii}(4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})s_{jj}^2.$$

In turn, this implies that

$$4\alpha_{11}\alpha_{22}s_{ii}^2 - \alpha_{ij}(\alpha_{ji} + \frac{1}{2}\gamma_{ji}^s\gamma_{ii}) = 4\alpha_{11}\alpha_{22}s_{ii} - \alpha_{ij}\alpha_{ji}s_{ii}.$$

Because $0 < s_{ii} < 1$, it follows from the latter that $\gamma_{ji}^s < 0$.

• In Section 4, just above Lemma 4.1., it is stated that in each point $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$, we have $t_{ii}(\gamma_{ij}, \gamma_{ji}) < u_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji}) < u_{ij}(\gamma_{ij}, \gamma_{ji})$. It is further stated there that in each $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ the slopes of the level curves of $u_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ii}(\gamma_{ij}, \gamma_{ji})$, as well as of $u_{ij}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji})$ are identical. Here we shall give the proof of these statements.

— Straightforward manipulations show that the elements of matrix U of (17) are given by

$$\begin{aligned} u_{ii} &= \frac{2\alpha_{jj}}{\det(R_0)} - \left[\frac{2\alpha_{jj} + \frac{1}{2}\gamma_{jj}^2}{\det(R)} \right] \\ &= \frac{2\alpha_{jj}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} - \left[\frac{2\alpha_{jj} + \frac{1}{2}\gamma_{jj}^2}{4\alpha_{11}\alpha_{22}s_{11}^2s_{22}^2 - (\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{21}\gamma_{11})} \right] \end{aligned} \quad (\text{A.6})$$

and

$$\begin{aligned} u_{ij} &= -\frac{\alpha_{ij}}{\det(R_0)} + \left[\frac{\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{ji}}{\det(R)} \right] \\ &= -\frac{\alpha_{ij}}{4\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} + \left[\frac{\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{ji}}{4\alpha_{11}\alpha_{22}s_{11}^2s_{22}^2 - (\alpha_{12} + \frac{1}{2}\gamma_{12}\gamma_{22})(\alpha_{21} + \frac{1}{2}\gamma_{21}\gamma_{11})} \right], \end{aligned} \quad (\text{A.7})$$

where $i, j = 1, 2$ ($j \neq i$).

Comparison of (A.6) and (A.7) with (A.1) and (A.2), respectively, directly shows that $t_{ii}(\gamma_{ij}, \gamma_{ji}) < u_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji}) < u_{ij}(\gamma_{ij}, \gamma_{ji})$.

— Next, observe that in all points of Γ_i there holds

$$\begin{aligned}\frac{\partial u_{ii}}{\partial \gamma_{ij}} &= \frac{-(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)\gamma_{jj}(\alpha_{ji} + \frac{1}{2}\gamma_{ji}\gamma_{ii})}{(\det(R))^2} > 0 \\ \frac{\partial u_{ii}}{\partial \gamma_{ji}} &= \frac{-(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})}{(\det(R))^2} > 0.\end{aligned}$$

Using this, the slope of the level curve of $u_{ii}(\gamma_{ij}, \gamma_{ji})$ in the point $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ is given by

$$\begin{aligned}\left(\frac{d\gamma_{ji}}{d\gamma_{ij}}\right)_{|u_{ii}=\bar{u}_{ii}} &= -\frac{\partial u_{ii}}{\partial \gamma_{ij}} / \frac{\partial u_{ii}}{\partial \gamma_{ji}} \\ &= -\frac{\gamma_{jj}(\alpha_{ji} + \frac{1}{2}\gamma_{ji}\gamma_{ii})}{\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})} \\ &= \left(\frac{d\gamma_{ji}}{d\gamma_{ij}}\right)_{|t_{ii}=\bar{t}_{ii}}\end{aligned}$$

where the last equality follows from (A.3). This shows that in each $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ the slopes of the level curves of $u_{ii}(\gamma_{ij}, \gamma_{ji})$ and $t_{ii}(\gamma_{ij}, \gamma_{ji})$ are identical.

— Finally, observe that in all points of Γ_i we have

$$\begin{aligned}\frac{\partial u_{ij}}{\partial \gamma_{ij}} &= \frac{2\gamma_{jj}(\alpha_{ii} + \frac{1}{4}\gamma_{ii}^2)(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)}{(\det(R))^2} > 0 \\ \frac{\partial u_{ij}}{\partial \gamma_{ji}} &= \frac{\frac{1}{2}\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})^2}{(\det(R))^2} > 0.\end{aligned}$$

As a result, the slope of the level curve of $u_{ij}(\gamma_{ij}, \gamma_{ji})$ in the point $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ is given by

$$\begin{aligned}\left(\frac{d\gamma_{ji}}{d\gamma_{ij}}\right)_{|u_{ij}=\bar{u}_{ij}} &= -\frac{\partial u_{ij}}{\partial \gamma_{ij}} / \frac{\partial u_{ij}}{\partial \gamma_{ji}} \\ &= -\frac{4\gamma_{jj}(\alpha_{ii} + \frac{1}{4}\gamma_{ii}^2)(\alpha_{jj} + \frac{1}{4}\gamma_{jj}^2)}{\gamma_{ii}(\alpha_{ij} + \frac{1}{2}\gamma_{ij}\gamma_{jj})^2} \\ &= \left(\frac{d\gamma_{ji}}{d\gamma_{ij}}\right)_{|t_{ij}=\bar{t}_{ij}}\end{aligned}$$

where the last equality follows from (A.4). This shows that in each $(\gamma_{ij}, \gamma_{ji}) \in \Gamma_i$ the slopes of the level curves of $u_{ij}(\gamma_{ij}, \gamma_{ji})$ and $t_{ij}(\gamma_{ij}, \gamma_{ji})$ are identical.